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Abstract: In this report we consider systems of ordinary differential equations depending on two parameters. We investigate a possibility of determination of stability zones of solutions by means of a computer. For solving this problem we use a method proposed by S.K.Godunov and that of by G.V.Demidenko for study of location of a matrix spectrum and for study of stability of solutions of systems of linear ordinary differential equations with constant coefficients. An applicability of these methods is investigated. This report contains some examples of computation of stability zones.

Key-words: matrix spectrum, stability in the sense of Lyapunov, asymptotic stability, Lyapunov equation

(Résumé : tsvp)

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Détermination des zones de stabilité pour des systèmes linéaires d'équations différentielles ordinaires dépendant de deux paramétres

Résumé : Dans ce rapport, nous considérons des systèmes d'équations différentielles ordinaires dépendant de deux paramétres. Nous recherchons une possibilité de déterminer sur ordinateur les zones de stabilité des solutions. Pour résoudre ce probléme nous utilisons une méthode proposée par S.K.Godounov et une autre proposée par G.V.Demidenko pour la localisation de spectres matriciels et pour étudier la stabilité des solutions des systèmes d'équations différentielles linéaires ordinaires à coefficients constants. Nous examinons comment s'appliquent ces méthodes. Ce rapport contient quelques exemples de calcul de zones de stabilité.

Mots-clé : spectre matriciel, stabilité au sens de Lyapounov, stabilité asymptotique, équation de Lyapounov

The aim of the present report is an investigation of stability of solutions of systems of ordinary differential equations with coefficients depending on two parameters by means of a method proposed by S.K.Godunov and that of by G.V.Demidenko. These methods were proposed for study of location of the spectrum of a constant matrix A and for study of stability of the solutions of the system of ordinary differential equations

$$\frac{dy}{dt} = Ay. \tag{1}$$

Let us consider the system of ordinary differential equations

$$\frac{dx}{dt} = A(v,e)x \tag{2}$$

where $A(v,e)$ is a matrix of order N depending on two parameters v, e. Many systems governing real physical processes are reduced to systems of the form (2). For example, consider the Lagrangian system (cf. [7])

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial y_k'} + e\frac{\partial \tilde{R}}{\partial y_k'} + \frac{\partial \tilde{K}}{\partial y_k} = vf_k, \quad k = 1,\dots,n, \tag{3}$$

where $\tilde{T}$ and $\tilde{R}$ are quadratic forms with respect to y_j', $\tilde{K}$ is a quadratic form with respect to y_j, f_k are linear functions with respect to y_j, $j = 1,\dots,n$. Indeed, (3) can be rewritten in the following form

$$Ty'' + eRy' + (K + vF)y = 0$$

or in the form (2), where

$$x = \begin{pmatrix} y \\ y' \end{pmatrix}, \qquad A(v,e) = \begin{pmatrix} 0 & I \\ -T^{-1}(K+vF) & -eT^{-1}R \end{pmatrix},$$

T, R, K, F are square matrices of order n and $det\,T \neq 0$.

Let us remind the definitions of stability in the sense of Lyapunov and the asymptotic stability of the solutions of the system (1) (see, e.g., [8]).

Definition. *A solution $y(t)$ of the system (1) is called stable in the sense of Lyapunov for $t > 0$ if for any $e > 0$ there exists $\delta(e) > 0$ such that $\|\tilde{y}(t) - y(t)\| \leq e$ for any $t \geq 0$ when $\|\tilde{y}(0) - y(0)\| \leq \delta(e)$.*

Definition. *A stable solution $y(t)$ of the system (1) is called asymptotically stable for $t > 0$ if there exists $\delta > 0$ such that $\|\tilde{y}(t) - y(t)\| \to 0$ for $t \to \infty$ when $\|\tilde{y}(0) - y(0)\| \leq \delta$.*

It is obvious that study of stability of solutions of systems of ordinary differential equations is reduced to study of stability of the null solution. Let us formulate the classical criteria of stability in the sense of Lyapunov and the asymptotic stability for the null solution of (1) [5, 8].

Theorem. *The null solution of the system (1) is stable in the sense of Lyapunov for $t > 0$ if and only if the spectrum of the matrix A belongs to the closed left half-plane $\{Re\,\lambda \leq 0\}$ and only one-dimensional Jordan blocks correspond to the imaginary eigenvalues.*

Theorem. *The null solution of the system (1) is asymptotically stable for $t > 0$ if and only if the spectrum of the matrix A belongs to the open left half-plane $\{Re\,\lambda < 0\}$.*

Thus, analytic investigation of stability of the solutions of (2) can be reduced to study of the spectrum of $A(v, e)$ for every fixed pair of the parameters (v, e). However, it is unreal to conduct investigation in general case. Therefore, it is necessary to have numerical methods which allow to solve this problem by means of a computer. But this gives rise to additional difficulties. Thus, calculation of eigenvalues of unsymmetric matrices by means of a computer is not often a stable process. Therefore, it is necessary also to have other approaches for study of stability of solutions.

The following Lyapunov theorem gives another approach for solving the problem of stability of the solutions of (1). It is not related to the eigenvalues of A.

Theorem. *The null solution of the system (1) for $t > 0$ is asymptotically stable if and only if the equation*

$$HA + A^*H = -C, \quad C = C^* > 0 \tag{4}$$

has a solution $H = H^* > 0$.

The equation (4) is called the Lyapunov equation. Note that if the spectrum of the matrix A belongs to the open left half-plane, then the unique solution of (4) has the form

$$\int_0^\infty e^{tA^*} C e^{tA} dt. \tag{5}$$

Thus, one can speak about the asymptotic stability of the solutions of (1) without calculation of the eigenvalues of A if the integral (5) converges.

Such approach to investigation of the asymptotic stability of the solutions of the system (1) by means of a computer was used by S.K.Godunov and A.Ya.Bulgakov (see, e.g., [1, 2, 6]). They suggested to consider the value $æ(A) = 2\|A\|\|H\|$ as a numerical characteristic of the asymptotic stability.

S.K.Godunov and A.Ya.Bulgakov elaborated an algorithm for computation of $æ(A)$ with a guaranteed accuracy. The algorithm permits to point out the systems of ordinary differential equations of the form (1) whose solutions are asymptotically stable with a guaranteed accuracy. This is equivalent to looking for the matrices whose spectra belong to the open left half-plane $\{Re\,\lambda < 0\}$. However, the problem of indicating the matrices whose spectra are located in the closed left half-plane $\{Re\,\lambda \leq 0\}$ has not been solved. To solve this problem and to investigate the problem of stability in the sense of Lyapunov for (1) S.K.Godunov and G.V.Demidenko proposed two different approaches.

At first, we present S.K.Godunov's suggestion. He proposed to consider the following sequence of the matrices

$$H(\rho_k) = \frac{\int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt}{tr\ \left(\int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt\right)}, \qquad k = 0, 1, \ldots$$

$$\text{where} \quad \rho_k \to \infty \qquad \text{for} \qquad k \to \infty,$$

C is a Hermitian positive definite matrix.

We now establish properties of the sequence $\{H(\rho_k)\}$.

Theorem 1. *The matrix*

$$H(\rho) = \frac{\int_0^{\rho} e^{tA^*} C\, e^{tA}\, dt}{tr\ \left(\int_0^{\rho} e^{tA^*} C\, e^{tA}\, dt\right)}, \qquad \rho > 0,$$

is Hermitian positive definite.

Proof. Let us determine the matrix

$$\tilde{H}(\rho) = \int_0^{\rho} e^{tA^*} C\, e^{tA}\, dt.$$

It is obvious that this matrix is Hermitian. Let us consider the quadratic form $\langle \tilde{H}(\rho)v, v\rangle$ for any vector v, $|v| \neq 0$. Then

$$\langle \tilde{H}(\rho)v, v\rangle = \langle \int_0^{\rho} e^{tA^*} C\, e^{tA}\, dt\, v, v\rangle$$

$$= \int_0^\rho \langle C\, e^{tA}v,\, e^{tA}v\rangle\, dt \geq \lambda_{min}(C) \int_0^\rho |\, e^{tA}v|^2\, dt.$$

As

$$|\, e^{tA}v| \geq\ e^{-|t|\|A\|}|v|,$$

then

$$\langle \tilde{H}(\rho)v, v\rangle \geq \lambda_{min}(C) \int_0^\rho\ e^{-2t\|A\|}\, dt\, |v|^2$$

$$= \lambda_{min}(C)\frac{1-\ e^{-2\rho\|A\|}}{2\|A\|}|v|^2 > 0.$$

Hence the matrix $\tilde{H}(\rho)$ is Hermitian positive definite. By the definition

$$H(\rho) = \frac{\tilde{H}(\rho)}{tr\, \tilde{H}(\rho)}.$$

As the trace of a positive definite matrix is positive, property 1 is proved.

Theorem 2. *There exists a sequence $\{\rho_k\}$, $\rho_k \to \infty$, $k \to \infty$ such that the sequence $\{H(\rho_k)\}$ converges when $k \to \infty$. The limit matrix is Hermitian nonnegative definite.*

Proof. By theorem 1 the matrix $H(\rho)$ is Hermitian positive definite. Hence, the following estimate holds

$$\frac{1}{N}\, tr\, H(\rho) \leq \|H(\rho)\| \leq tr\, H(\rho).$$

Since $tr\, H(\rho) = 1$, it follows that

$$\frac{1}{N} \leq \|H(\rho)\| \leq 1,$$

i. e. the family $\{H(\rho)\}$, $\rho \geq \rho_0 > 0$ belongs to the bounded closed set $\{M \in \mathbf{M}_N :\ \frac{1}{N} \leq \|M\| \leq 1\}$ in the space $\mathbf{M}_N$ of the matrices of order N. This set is compact because the space $\mathbf{M}_N$ is finite-dimensional. Consequently, there exists a sequence $\{\rho_k\}$, $\rho_k \to \infty$, $k \to \infty$, such that the sequence $\{H(\rho_k)\}$ converges. By theorem 1 the limit matrix is Hermitian nonnegative definite. Note that theorem 2 doesn't state that the sequence $\{H(\rho_k)\}$ has the limit for any sequence $\{\rho_k\}$, $\rho_k \to \infty$, $k \to \infty$. We now give a number of important examples of matrices such that the limit exists for any sequence $\{\rho_k\}$.

Theorem 3. *If the spectrum of the matrix A belongs to the open left half-plane $\{Re\,\lambda < 0\}$, then for $\rho_k \to \infty$, $k \to \infty$, there exists the limit*

$$H = \lim_{k\to\infty} H(\rho_k) \tag{6}$$

and the limit matrix satisfies the Lyapunov equation

$$HA + A^*H = -\left(tr\int_0^\infty e^{tA^*}Ce^{tA}dt\right)^{-1} C.$$

Theorem 4. *Let the spectrum of the matrix A belong to the closed left half-plane $\{Re\,\lambda \leq 0\}$. Suppose that only one-dimensional Jordan blocks correspond to the imaginary eigenvalues, then for $\rho_k \to \infty$, $k \to \infty$, there exists the limit (6).*

Theorem 5. *Suppose that one eigenvalue of the matrix A lies strictly in the right half-plane and the remaining eigenvalues belong to the open left half-plane, then for $\rho_k \to \infty$, $k \to \infty$, there exists the limit (6).*

Theorem 6. *Suppose that one eigenvalue of the matrix A lies strictly in the right half-plane and the remaining eigenvalues belong to the closed left half-plane. If only one-dimensional Jordan blocks correspond to the imaginary eigenvalues, then for $\rho_k \to \infty$, $k \to \infty$, there exists the limit (6).*

Let us prove these theorems.

Proof of theorem 3. If the spectrum of the matrix A belongs to the open left half-plane then the integral

$$\tilde{H} = \int_0^\infty e^{tA^*}C\,e^{tA}\,dt \tag{7}$$

exists and

$$H = \lim_{k\to\infty} H(\rho_k) = \lim_{k\to\infty}\int_0^{\rho_k} e^{tA^*}C\,e^{tA}\,dt$$

$$\times \lim_{k\to\infty}\frac{1}{tr\left(\int_0^{\rho_k} e^{tA^*}C\,e^{tA}\,dt\right)} = \frac{\int_0^\infty e^{tA^*}C\,e^{tA}\,dt}{tr\left(\int_0^\infty e^{tA^*}C\,e^{tA}\,dt\right)}.$$

As already mentioned, the integral (7) is the solution of the Lyapunov equation (4).

Theorem 3 is proved.

To prove theorems 4 - 6 we need the following

Lemma. *Let $C = C^* > 0$ and $J = T^{-1}AT$, then there exist constants $d_1,\ d_2 > 0$ such that for any $\rho \geq \rho_0 > 0$ the following estimate holds*

$$d_2\ tr\left(\int_0^\rho e^{tJ^*}e^{tJ}dt\right) \leq tr\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right) \leq d_1\ tr\left(\int_0^\rho e^{tJ^*}e^{tJ}dt\right). \quad (8)$$

Proof. From the conditions of the lemma

$$e^{tA} = Te^{tJ}T^{-1}.$$

Hence, for any vector v we have

$$\begin{aligned}\langle\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)v, v\rangle &= \langle (T^{-1})^*\left(\int_0^\rho e^{tJ^*}(T^*CT)e^{tJ}dt\right)T^{-1}v, v\rangle \\ &= \langle\left(\int_0^\rho e^{tJ^*}(T^*CT)e^{tJ}dt\right)T^{-1}v, T^{-1}v\rangle \\ &= \int_0^\rho \langle (T^*CT)e^{tJ}(T^{-1}v), e^{tJ}(T^{-1}v)\rangle dt.\end{aligned}$$

Since the matrix T^*CT is Hermitian positive definite, it follows that there exist constants $a_1,\ a_2 > 0$ such that

$$a_2\|e^{tJ}(T^{-1}v)\|^2 \leq \langle (T^*CT)e^{tJ}(T^{-1}v), e^{tJ}(T^{-1}v)\rangle \leq a_1\|e^{tJ}(T^{-1}v)\|^2.$$

Therefore, for any $\rho > 0$ we obtain

$$a_2\int_0^\rho \|e^{tJ}(T^{-1}v)\|^2dt \leq \langle\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)v, v\rangle \leq a_1\int_0^\rho \|e^{tJ}(T^{-1}v)\|^2dt.$$

One can rewrite these inequalities in the form

$$\begin{aligned}a_2\langle\left(\int_0^\rho e^{tJ^*}e^{tJ}dt\right)T^{-1}v, T^{-1}v\rangle &\leq \langle\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)v, v\rangle \\ &\leq a_1\langle\left(\int_0^\rho e^{tJ^*}e^{tJ}dt\right)T^{-1}v, T^{-1}v\rangle.\end{aligned}$$

Since the matrix $\int_0^\rho e^{tJ^*}e^{tJ}dt$ is Hermitian positive definite it follows that there exist constants $b_1,\ b_2 > 0$ being independent on ρ such that

$$b_2\,tr\left(\int_0^\rho e^{tJ^*}e^{tJ}dt\right)\|T^{-1}v\|^2 \leq \langle\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)v, v\rangle$$

$$\leq b_1\, tr\left(\int_0^\rho e^{tJ^*}e^{tJ}dt\right)\|T^{-1}v\|^2.$$

Since the matrix $\int_0^\rho e^{tA^*}Ce^{tA}dt$ is also Hermitian positive definite, then there exist constants $c_1,\ c_2 > 0$ being independent on ρ such that

$$c_2 tr\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)\|v\|^2 \leq \langle\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)v,v\rangle$$

$$\leq c_1 tr\left(\int_0^\rho e^{tA^*}Ce^{tA}dt\right)\|v\|^2.$$

The estimate (8) can then be obtained as a consequence of these inequalities.

Lemma is proved.

Let us explain the idea of the proofs of theorems 4-6.

At first, put the matrix A into the Jordan canonical form

$$J = T^{-1}AT. \tag{9}$$

From the lemma we obtain that for the proof of existence of the limit (6) it is sufficient to show existence of the limit

$$\lim_{k\to\infty}\frac{\int_0^{\rho_k} e^{tA^*}Ce^{tA}dt}{tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right)}. \tag{10}$$

This essentially simplifies the proof of the convergence (6) because it is not difficult to calculate the trace

$$tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right). \tag{11}$$

Note that we can prove the convergence (10) if there exists the limit

$$\lim_{k\to\infty}\frac{\langle\left(\int_0^{\rho_k} e^{tA^*}Ce^{tA}dt\right)u,v\rangle}{tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right)} \tag{12}$$

for any vectors $u,\ v$. But this is equivalent to the convergence

$$\lim_{k\to\infty}\frac{\langle\left(\int_0^{\rho_k} e^{tJ^*}T^*Te^{tJ}dt\right)u,v\rangle}{tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right)}. \tag{13}$$

Proof of theorem 4. By the conditions of the theorem the matrix A can be put into the Jordan canonical form (9) where

$$J = \begin{pmatrix} J_- & 0 \\ 0 & J_0 \end{pmatrix},$$

J_- is a matrix with the Jordan blocks on the diagonal and these blocks correspond to the eigenvalues with negative real parts,

$$J_0 = \begin{pmatrix} i\eta_1 & & 0 \\ & \ddots & \\ 0 & & i\eta_q \end{pmatrix},$$

$i\eta_1, \dots, i\eta_q$ are the imaginary eigenvalues of A.

It is obvious that

$$e^{tJ^*} e^{tJ} = \begin{pmatrix} e^{tJ_-^*} e^{tJ_-} & 0 \\ 0 & e^{tJ_0^*} e^{tJ_0} \end{pmatrix}$$

where the elements of the matrix $e^{tJ_-^*} e^{tJ_-}$ are linear combinations of summands of the form $p_m(t)e^{t\lambda_j}$, p_m are polynomials of degree m with respect to t, $Re\, \lambda_j < 0$ and

$$e^{tJ_0^*} e^{tJ_0} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

From the positive definiteness of $e^{tJ^*} e^{tJ}$ we obtain that $tr\left(\int_0^{\rho_k} e^{tJ^*} e^{tJ} dt\right)$ has the form

$$\sum_j \int_0^{\rho_k} p_{m_j}(t) e^{t\alpha_j} dt + q\rho_k$$

where $\alpha_j < 0$, $p_{m_j} > 0$. Hence, if $\rho_k \to \infty$, $k \to \infty$, then

$$\frac{tr\,(\int_0^{\rho_k} e^{tJ^*} e^{tJ} dt)}{\rho_k} \to q. \tag{14}$$

We now show that there exists the limit (13).

Taking into account the structure of e^{tJ} we have

$$e^{tJ^*}T^*Te^{tJ} = \begin{pmatrix} K(t) & L(t) \\ M(t) & N(t) \end{pmatrix} \tag{15}$$

where the elements of $K(t)$, $L(t)$, $M(t)$ have the form $p_m(t)e^{t(\alpha+i\beta)}$, $\alpha < 0$, and the elements of $N(t)$ have the form $p_0 e^{i\beta t}$. Since for $k \to \infty$

$$\frac{1}{\rho_k}\int_0^{\rho_k} p_m(t)e^{t(\alpha+i\beta)}dt \to 0, \quad \text{for} \quad \alpha < 0,$$

$$\frac{1}{\rho_k}\int_0^{\rho_k} p_0 e^{i\beta t}dt \to 0, \quad \text{for} \quad \beta \neq 0,$$

$$\frac{1}{\rho_k}\int_0^{\rho_k} p_0 e^{i\beta t}dt = p_0, \quad \text{for} \quad \beta = 0,$$

then from (14) we obtain the convergence (13).

The theorem is proved.

Proof of theorem 5. By the conditions of the theorem the matrix A can be put into the Jordan canonical form (9) where

$$J = \begin{pmatrix} J_- & 0 \\ 0 & J_+ \end{pmatrix},$$

J_- is a matrix with the Jordan blocks on the diagonal and these blocks correspond to the eigenvalues with negative real parts,

$$J_+ = \begin{pmatrix} J_+^1 & & 0 \\ & \ddots & \\ 0 & & J_+^q \end{pmatrix}$$

where J_+^k is the Jordan blocks of order k, $k = 1, \dots, q$, corresponding to the eigenvalues λ^+ with $Re\,\lambda^+ > 0$.

Let J_+^n is a Jordan block of order l_n. Then

$$e^{J_+^n} = e^{t\lambda^+}\begin{pmatrix} 1 & \frac{t}{1!} & \cdots & \frac{t^{l_n-1}}{(l_n-1)!} \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

This leads to

$$e^{t(J_+^n)^*}e^{tJ_+^n} = e^{2tRe\,\lambda^+}(p_{ij}(t))$$

where

$$\begin{aligned}
p_{11}(t) &= 1, \\
p_{22}(t) &= \left(\frac{t}{1!}\right)^2 + 1, \\
&\vdots \\
p_{l_n l_n}(t) &= \left(\frac{t^{l_n-1}}{(l_n-1)!}\right)^2 + \ldots + \left(\frac{t}{1!}\right)^2 + 1,
\end{aligned}$$

and for $i \neq j$ $p_{ij}(t)$ are polynomials whose degrees are less than or equal to $(2l_n - 3)$. Since

$$e^{tJ^*}e^{tJ} = \begin{pmatrix} e^{tJ_-^*}e^{tJ_-} & 0 \\ 0 & e^{tJ_+^*}e^{tJ_+} \end{pmatrix},$$

it follows that

$$tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right) = \sum_j \int_0^{\rho_k} p_{m_j}(t)e^{t\alpha_j}dt$$

$$+\sum_{n=1}^{q}\int_0^{\rho_k}\left(1 + \left[\left(\frac{t}{1!}\right)^2 + 1\right] + \ldots + \left[\left(\frac{t^{l_n-1}}{(l_n-1)!}\right)^2 + \ldots + 1\right]\right)e^{2tRe\,\lambda^+}dt$$

where $\alpha_j < 0$, $p_{m_j}(t) > 0$ are polynomials.

Let $l_{max} = max(l_1, \ldots, l_q)$. Since $Re\,\lambda^+ > 0$ then

$$\frac{tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right)}{\int_0^{\rho_k}\left(\frac{t^{l_{max}-1}}{(l_{max}-1)!}\right)^2 e^{2tRe\,\lambda^+}dt} \to r \tag{16}$$

where r is the number of the Jordan blocks J_+^i whose order is equal to l_{max}.

We now show that there exists the limit (13).

Taking into account the structure of e^{tJ} we have (15), where the elements of $K(t)$ have the form $p_k(t)e^{t(\alpha+i\beta)}$, $\alpha < 0$, the elements of $L(t)$, $M(t)$ have the form $r_m(t)e^{t(\lambda^+ + \alpha + i\beta)}$ and the elements of $N(t)$ have the form $q_n e^{2t\,Re\,\lambda^+}$,

$p_k(t)$, $r_m(t)$ and $q_n(t)$ are polynomials and degrees of $q_n(t)$ are less than or equal to $(2l_{max}-2)$. Since for $0 \le s < 2l_{max}-3$

$$\frac{\int_0^{\rho_k} t^s e^{2t\,Re\,\lambda^+}dt}{\int_0^{\rho_k} t^{2l_{max}-2}e^{2tRe\,\lambda^+}dt} \to 0, \quad \text{for} \quad \rho_k \to \infty$$

and for $\alpha < 0$

$$\frac{\int_0^{\rho_k} p_k(t)e^{t(\alpha+i\beta)}dt}{\int_0^{\rho_k} t^{2l_{max}-2}e^{2tRe\,\lambda^+}dt} \to 0, \quad \text{for} \quad \rho_k \to \infty,$$

$$\frac{\int_0^{\rho_k} r_m(t)e^{t(\lambda^+ +\alpha+i\beta)}dt}{\int_0^{\rho_k} t^{2l_{max}-2}e^{2tRe\,\lambda^+}dt} \to 0, \quad \text{for} \quad \rho_k \to \infty,$$

then by (16) we have the convergence (13).

Theorem is proved.

The proof of theorem 6 can be obtained similarly to those of theorems 4 and 5.

Theorem 7. *Let the spectrum of the matrix A belong to the closed left half-plane and there exists at least one imaginary eigenvalue. If the limit (6) exists, then the limit matrix H satisfies the relation*

$$HA + A^*H = 0.$$

Theorem 8. *Let a part of the spectrum of the matrix A lie in the right half-plane. If the limit (6) exists, then the limit matrix H satisfies the relation*

$$HA + A^*H = D,$$

where D is a Hermitian nonnegative definite matrix.

Proofs of theorems 7 and 8. By analogy with the proofs of theorems 4 and 5, we obtain that if the spectrum of the matrix A doesn't belong to the open left half-plane, then

$$tr\left(\int_0^{\rho_k} e^{tJ^*}e^{tJ}dt\right) \to \infty, \quad \rho_k \to \infty, \quad k \to \infty.$$

Hence, from the lemma

$$tr\left(\int_0^{\rho_k} e^{tA^*}Ce^{tA}dt\right) \to \infty, \quad \rho_k \to \infty, \quad k \to \infty.$$

This leads to

$$HA + A^*H = \lim_{k\to\infty} \left(\frac{\int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt\, A}{tr(\int_0^{\rho_k} e^{tA^*} C\, e^{tA} dt)} \right.$$

$$\left. + \frac{A^* \int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt}{tr(\int_0^{\rho_k} e^{tA^*} C\, e^{tA} dt)} \right) = \lim_{k\to\infty} \frac{\int_0^{\rho_k} \frac{d}{dt}(e^{tA^*} C\, e^{tA})\, dt}{tr(\int_0^{\rho_k} e^{tA^*} C\, e^{tA} dt)}$$

$$= \lim_{k\to\infty} \frac{e^{\rho_k A^*} C\, e^{\rho_k A} - I}{tr(\int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt)} = \lim_{k\to\infty} \frac{e^{\rho_k A^*} C\, e^{\rho_k A}}{tr(\int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt)} = D,$$

where D is a Hermitian nonnegative definite matrix.

We now show that $D = 0$ if the spectrum of A satisfies the conditions of theorem 7.

Indeed, let p be the maximal size of the Jordan blocks corresponding to the imaginary eigenvalue λ_j and v_p be a corresponding generalized vector

$$e^{tA} v_p = e^{\lambda_j t} \left(\frac{t^{p-1}}{(p-1)!} v_1 + \ldots + v_p \right).$$

Let us consider the case of $p > 1$ ($p = 1$ by analogy). Then

$$\frac{\| e^{\rho_k A^*} C\, e^{\rho_k A} \|}{tr\,(\int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt)} \le \|C\| \frac{\| e^{\rho_k A}\|^2}{\| \int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt\|}$$

$$\le c_1 \frac{(1 + \rho_k \|A\|)^{2(p-1)}}{\langle \int_0^{\rho_k} e^{tA^*} C\, e^{tA}\, dt\, v_p, v_p \rangle} |v_p|^2 \le c_2 \frac{(1 + \rho_k \|A\|)^{2(p-1)}}{\int_0^{\rho_k} |\, e^{tA} v_p|^2\, dt} |v_p|^2$$

$$= c_2 \frac{(1 + \rho_k \|A\|)^{2(p-1)}}{\int_0^{\rho_k} \left| \frac{t^{p-1}}{(p-1)!} v_1 + \ldots + v_p \right|^2 dt} |v_p|^2$$

(according to the Minkovskii inequality)

$$\le c_2 \frac{(1 + \rho_k \|A\|)^{2(p-1)}}{\left| \int_0^{\rho_k} \left| \frac{t^{p-1}}{(p-1)!} v_1 \right|^2 dt - \int_0^{\rho_k} \sum_{j=0}^{p-2} \left| \frac{t^j}{j!} v_{p-j} \right|^2 dt \right|} |v_p|^2$$

$$\le c_2 \frac{(1 + \rho_k \|A\|)^{2(p-1)}}{|d_{p-1} \rho_k^{2p-1} - \sum_{j=0}^{p-2} d_j \rho_k^{2j+1}|} |v_p|^2$$

$$= \frac{c_2}{\rho_k} \frac{\left(\frac{1}{\rho_k} + \|A\|\right)^{2p-2}}{|d_{p-1} - \sum_{s=-2}^{-2p+2} d_j \rho_k^s|} |v_p|^2 \to 0 \quad \text{for} \quad k \to \infty.$$

Theorem 7 and 8 are proved.

Example 1.

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha < 0, \quad \beta < 0, \qquad C = I,$$

$$H = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & 0 \\ 0 & \frac{\alpha}{\alpha+\beta} \end{pmatrix}, \qquad HA + A^*H = \begin{pmatrix} \frac{2\alpha\beta}{\alpha+\beta} & 0 \\ 0 & \frac{2\alpha\beta}{\alpha+\beta} \end{pmatrix}.$$

Example 2.

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha < 0, \quad \beta = i\gamma, \qquad C = I,$$

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad HA + A^*H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 3.

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha = i\gamma, \quad \beta > 0, \qquad C = I,$$

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad HA + A^*H = \begin{pmatrix} 0 & 0 \\ 0 & 2\beta \end{pmatrix}.$$

Example 4.

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0, \quad \beta > 0, \quad \beta > \alpha, \qquad C = I,$$

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad HA + A^*H = \begin{pmatrix} 0 & 0 \\ 0 & 2\beta \end{pmatrix}.$$

Example 5.

$$A = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad \beta \in R_1, \qquad C = I,$$

$$H=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad HA+A^*H=\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}.$$

Example 6.

$$A=\begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}, \quad \beta \geq 0, \qquad C=I,$$

$$H=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad HA+A^*H=\begin{pmatrix} 0 & 0 \\ 0 & 2\beta \end{pmatrix}.$$

Example 7.

$$A=\begin{pmatrix} \alpha+i\beta & 0 \\ 0 & \alpha+i\beta \end{pmatrix}, \quad \alpha \geq 0, \qquad C=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

a). If $\rho_k=\frac{\pi k}{|\beta|}$, then

$$H=\lim_{\rho_k\to\infty} H(\rho_k)=\frac{1}{c_{11}+c_{22}}\begin{pmatrix} c_{11} & \frac{\alpha}{\alpha-i\beta}c_{12} \\ \frac{\alpha}{\alpha+i\beta}c_{21} & c_{22} \end{pmatrix}.$$

b). If $\rho_k=\frac{\pi/2+2\pi k}{2|\beta|}$, then

$$H=\lim_{\rho_k\to\infty} H(\rho_k)=\frac{1}{c_{11}+c_{22}}\begin{pmatrix} c_{11} & \frac{-i\alpha}{\alpha-i\beta}c_{12} \\ \frac{i\alpha}{\alpha+i\beta}c_{21} & c_{22} \end{pmatrix}.$$

This method allows to pick out the matrices whose spectra belong to the closed left half-plane. However, it gives no information about the number and the multiplicity of the imaginary eigenvalues. The latter is essential for investigation of stability questions. Thus, in example 2 the matrix A has one negative eigenvalue and one imaginary eigenvalue, i.e. the solutions of the system $\frac{dy}{dt}=Ay$ are stable for $t>0$. In example 6 for $\beta=0$ the matrix A has two null eigenvalues, moreover, the two-dimensional Jordan block corresponds to these eigenvalues, i.e. the solutions of the system $\frac{dy}{dt}=Ay$ are unstable. But if we will investigate the spectra of these matrices by means of the present method, we can say only that A has at least one eigenvalue on the imaginary axis both in example 2 and in example 6. We can say nothing about stability of the solutions of the corresponding systems of ordinary differential equations.

Another approach for solving the problem of picking out the matrices whose spectra are located in the closed left half-plane $\{Re\,\lambda \leq 0\}$ was proposed by G.V.Demidenko [3, 4]. This approach is not connected with calculation of eigenvalues and doesn't use the Lyapunov equation. It permits to establish more rigorous computational results than the parameter æ(A) under studing the asymptotic stability. It gives also a method of study of stability in the sense of Lyapunov for solutions of (1).

We now describe in brief the main idea of this method.

G.V.Demidenko introduced a family of spectral characteristics $æ_p(A)$, $p \geq 0$ for the matrix A. He determined the parameter $æ_p(A)$ as follows: if there exists the integral

$$H_p = \int_0^\infty (1 + t\|A\|)^{-2p} e^{tA^*} e^{tA} dt, \tag{17}$$

then

$$æ_p(A) = a_p \|A\| \|H_p\| \tag{18}$$

where

$$a_p = \left(\int_0^\infty (1+s)^{-2p} e^{-2s} ds \right)^{-1}.$$

If the integral (17) diverges, then $æ_p(A) = \infty$.

From the definition of the integral (17) we have that $æ_0(A)$ coincides with æ(A) for the Hurwitz matrices. Indeed, if the spectrum of A belongs to the open left half-plane, then H_0 is the solution of the Lyapunov equation (4) with $C = I$. However, if H_p exists for $p > 0$, then it doesn't satisfy the Lyapunov equation. In this case there exists the matrix $H_{p+1/2}$ and

$$H_p A + A^* H_p = -I + 2p\|A\| H_{p+1/2}.$$

The detailed proof can be found in [3].

We now formulate some results from [3, 4].

First, we list some properties of the matrices (17).

1. *The matrix H_p is Hermitian positive definite.*

2. *If the matrix H_p exists, then for $q > p$ the matrix H_q also exists and*

$$\|H_p\| > \|H_q\|.$$

3. *If for some p the matrix H_p exists, then all eigenvalues of A belong to the closed left half-plane $\{Re\,\lambda \leq 0\}$.*

4. *If A has at least one eigenvalue with positive real part, then for all $p \geq 0$ the matrix H_p is indefinite.*

5. *If the spectrum of A belongs to the closed left half-plane $\{Re\,\lambda \leq 0\}$, then the matrix H_N is definite.*

The following properties of the characteristics $æ_p(A)$ ensue from their definition (18) and the properties of H_p.

1^0. $æ_p(A) = a_p\|A\| \max_{\|v\|=1} \left(\int_0^\infty (1 + s\|A\|)^{-2p}\|e^{sA}v\|^2 ds\right)$.

2^0. *If $æ_p(A) < \infty$, then $æ_q(A) < æ_p(A)$ for $q > p$.*

3^0. *If $æ_p(A) < \infty$ for certain $p \geq 0$, then all eigenvalues of A have nonpositive real parts.*

4^0. *If A has at least one eigenvalue with positive real part, then $æ_p(A) = \infty$ for all $p \geq 0$.*

5^0. *If all eigenvalues of A have nonpositive real parts, then there exists certain $0 \leq p \leq N$ such that $æ_p(A) < \infty$.*

6^0. $æ_p(A) = æ_p(A/\|A\|)$.

There is a preliminary algorithm for calculation of $æ_p(A)$ in [3]. This algorithm demonstrates a principal possibility of calculation of $æ_p(A)$ by means of a computer. However, at present, there is no algorithm for calculation of these characteristics with a quaranteed accuracy yet. Note that property 6^0 allows to calculate $æ_p$ for matrices with the unit norm.

The following spectral criterion for A ensue from properties 3^0 - 5^0.

Theorem 9. *The spectrum of A belongs to the closed left half-plane if and only if $æ_N(A) < \infty$.*

The spectral characteristics $æ_p(A)$ allow to introduce new criteria of the asymptotic stability and stability in the sense of Lyapunov for the solutions of the system (1). We now formulate two criteria [4].

Theorem 12. *The null solution of the system (1) is asymptotically stable for $t > 0$ if and only if $æ_p(A) < \infty$ for some $0 < p \leq 1/2$.*

Theorem 13. *The null solution of the system (1) is stable in the sense of Lyapunov for $t > 0$ if and only if $æ_0(A) = \infty,\ æ_1(A) < \infty$.*

According to theorem 12, one can use the parameter $æ_p(A)$, $0 < p \leq 1/2$ for study of the asymptotic stability. Note that by property 2^0 $æ_p(A) < æ_0(A) = æ(A)$. Therefore, one can obtain more rigorous rezults on a computer, using the parameter $æ_p(A)$, $0 < p \leq 1/2$ instead of $æ(A)$ for determination of the asymptotic stability zones in ranges of variations of parameters. By theorem 13 one can indicate the stability zones computating the parameter $æ_1(A)$.

Consider the system

$$z'' + e\hat{R}z' + (\hat{K} + v\hat{F})z = 0,$$

$$\hat{K} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} -1 & 1 \\ -2.75 & 3 \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

or

$$\frac{dy}{dt} = Ay,$$

$$y = \begin{pmatrix} z \\ z' \end{pmatrix}, \quad A = A(v,e) = \begin{pmatrix} 0 & I \\ -\hat{K} - v\hat{F} & -e\hat{R} \end{pmatrix}.$$

It is easy to calculate that the zone of asymptotic stability is

$$\Omega_1 = \left\{(v,e) : \quad \frac{(v-1)^2}{1/4} + \frac{e^2}{2/5} < 1,\ e > 0\right\},$$

i.e. the internal part of half-ellipse. And the zone of stability is

$$\Omega_2 = \left\{(v,e) : \quad \frac{(v-1)^2}{1/4} + \frac{e^2}{2/5} \leq 1,\ e \geq 0\right\} \cup \{(v,e) : \quad v \geq -1\},\ e = 0\}.$$

On Figure 1 one can see the asymptotic stability zone computated with the help of the characteristic $æ_0(A)$, on Figure 2 - the asymptotic stability zone and the stability zone computated with the help of the characteristic $æ_1(A)$.

point-types	**order of $æ_0(A)$ ($æ_1(A)$)**
.	10^1
o	10^2
*	10^3
+	10^4
x	10^5

The present investigations were conducted at the Institute of Mathematics, Siberian Branch of Russian Academy of Sciences, Novosibirsk, Russia and at IRISA, Rennes, France.

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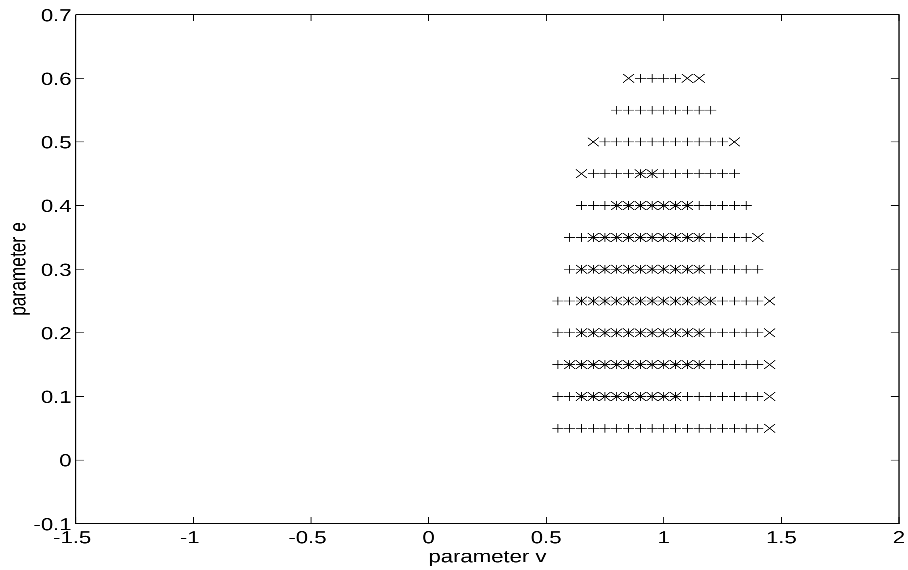

Figure 1: Asymptotic stability zone

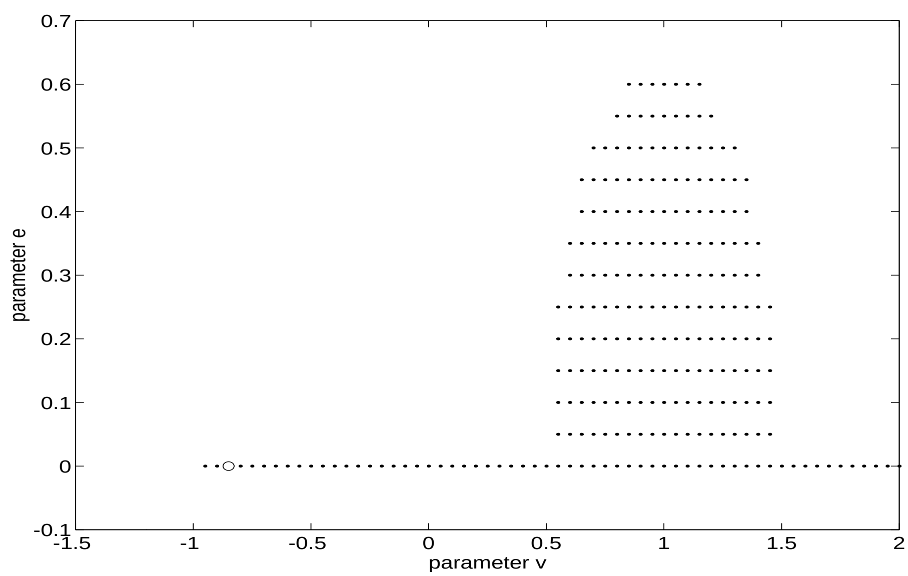

Figure 2: Asymptotic stability zone and stability zone

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